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Yang–Mills Gauge Theory and Gravitation

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We point out that Yang's and Einstein's gravitational equations can be obtained from a geometric approach of Yang–Mills gauge theory in a sourceless case, under a decomposition of the Poincaré algebra. Otherwise, Einstein's equations cannot be derived from a Yang–Mills gauge equation when sources are inserted in the rotational sector of that algebra. A gauge Lagrangian structure is also discussed.

1. INTRODUCTION

Analogies between the Yang–Mills (YM) theory at the classical level and general relativity (GR), under their common basic geometrical setting, have long been studied. Indeed, if we look for a spacetime gauge model for gravitation, it is necessary to investigate the features of spacetime gauge-like characteristics: on any differentiable manifold there is a bundle of affine frames, naturally defined, whose structural group is the affine linear group $AL(n, R) = GL(n, R) \otimes T_n$. For the spacetime case, in particular, the requirement of Lorentz frames reduces $AL(n, R)$ to the Poincaré group $P_4 = SO(3, 1) \varnothing T_4$. Gauge theories for the Poincaré and de Sitter groups have been extensively studied as alternatives theories for gravitation.

The local geometrical structure of GR as a gauge theory for the de Sitter group $SO(3, 2)$ has already been analyzed in detail (Stelle and West, 1980). To reproduce the structure of Einstein–Cartan theory the $SO(3, 2)$ gauge symmetry is spontaneously broken down to the Lorentz group. In such an approach the gravitational vierbein and spin connections can be derived from their original $SO(3, 2)$ gauge fields by passing over to a set of nonlinearly transforming fields, through a redefinition involving a Goldstone field. Moreover, the original $SO(3, 2)$ gauge fields generate pseudotranslations and

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rotations in the so-called internal anti-de Sitter space under a kind of parallel transport. Norris *et al.* (1980) proposed an underlying fiber bundle structure for gauge theories of gravitation and an extension to an affine structure group. They pointed out that the decomposition of a generalized affine connection contains more than curvature and torsion. This also should affect the field equations. Moreover, they considered an extension of the linear frame bundle to the affine frame bundle and established that the torsion is just one part of the *total curvature* of a generalized affine connection.

Within the framework of differential geometry Mielke (1981, 1986) considered a Yang parallel displacement gauge theory related to pure gravitational fields. He showed that, in a four-dimensional Riemannian manifold, double self-dual solutions obey Einstein's vacuum equation with a cosmological term, whereas the double anti-self-dual configurations satisfy the Raynich conditions of geometrodynamics. Moreover, under duality conditions the Stephenson–Kilmister–Yang theory not only embraces Einstein's sourceless equations $R_{\mu\nu} = 0$, but also Nordstrom's vacuum theory. We recall that the Lagrangian structure of the Poincaré gauge field equations for gravitation, and their Einsteinian content, under duality conditions is already known (Aldrovandi and Stedile, 1984). However, there has been much criticism concerning a YM gravitational model for the Poincaré group. A main point frequently made is that GR does not have the entire Poincaré local symmetry of spacetime and also, since the Poincaré group is not semisimple, we cannot build up a Lagrangian theory for this group. Besides, the essential fact still remains: the Einstein–Hilbert Lagrangian is not of the YM type, and the dynamical aspects of both theories are qualitatively different.

2. FIELD EQUATIONS

We start by considering a gauge model in a principal bundle $P = (M, G)$, where M is the base-manifold (Minkowski spacetime) and $G = SO(3, 1)) \varnothing T_4$ is the Poincaré group. Here the $\mathcal{G}$-algebra of G is a vector space, given by the direct sum $\mathcal{G} = \mathcal{R} \oplus \mathcal{T}$, where $\mathcal{R}$ and $\mathcal{T}$, are respectively, the rotational and translational sectors of $\mathcal{G}$. In a basis with generators $[J_{ab}, I_c]$ an affine connection $\overline{\Gamma}$ on the P-bundle decomposes into $\overline{\Gamma} = \Gamma + S$, where $\Gamma = J^b{}_a \Gamma^a{}_{b\mu}\, dx^\mu$ is an $\mathcal{R}$-valued connection and $S = I_c h^c{}_\lambda\, dx^\lambda$ is the $\mathcal{T}$-valued solder form. Here both forms are written in a coordinate basis $\{dx^\mu\}$ in spacetime.

Such a decomposition affects the curvature of $\overline{\Gamma}$

$$\overline{F} = F + T \tag{1}$$

where F and T are the curvature and the torsion of Γ, respectively:

$$F = d\Gamma + \Gamma \wedge \Gamma, \qquad T = dS + \Gamma \wedge S + S \wedge \Gamma \tag{2}$$

The above decomposition of the Lie algebra $\mathcal{G}$ gives rise to the Bianchi identities

$$dF + [\Gamma, F] = 0, \qquad dT + [\Gamma, T] + [S, F] = 0 \tag{3}$$

Yang–Mills equations are here written as a breaking of dual symmetry of Bianchi identities for any group, once its structure constants are known. Also, if sources are to be inserted in such equations, these sources should be the Noether current densities, whose "charges" are the generators of the symmetry group. At first sight, they could be the density of the relativistic angular momentum Λ and the stress-energy Θ:

$$d * F + [\Gamma, *F] = *\Lambda, \qquad d * T + [\Gamma, *T] + [S, *F] = *\Theta \tag{4}$$

Here we notice that the torsion is always present in the bundle of frames, and its vanishing should lead to GR. Also, the last equation points out a propagation for the torsion field, and since it is a dynamical equation, it is different from Einstein–Cartan equation (Hehl, 1979).

To obtain the Riemannian limit of equations (4), we notice that these equations may be projected onto the base manifold by means of the four-legs $h^a{}_\alpha$. Moreover, locally the base manifold is endowed with a Riemannian structure when we consider a Levi-Civitá connection Γ:

$$d * F + [\Gamma, *F] = *\Lambda, \qquad [S, *F] = *\Theta \tag{5}$$

which lead respectively to

$$\partial^\lambda \tilde{F}^a_{b\mu\lambda} + \Gamma^a_c{}^\lambda \tilde{F}^c_{b\mu\lambda} - \Gamma^c_b{}^\lambda \tilde{F}^a_{c\mu\lambda} = \Lambda^a{}_{b\mu} \tag{6}$$

$$S^b{}_\lambda \tilde{F}^a_{b\mu}{}^\lambda = \Theta^a{}_\mu \tag{7}$$

where the tilde stands for the dual. In reality Γ can be defined in many different ways, with S being a horizontal form, too, of a more general type. The solder form is particularly convenient because, when written in a frame given by the four-leg field h^a_α, its components are those of the dual basis (see, for instance, Kobayashi and Nomizu, 1963). In the dual basis the above equations become, respectively, in a Riemannian spacetime

$$\nabla^\lambda \tilde{R}^\alpha{}_{\beta\mu\lambda} = \Lambda^\alpha{}_{\beta\mu}, \qquad \tilde{R}^\alpha{}_\beta{}^\beta{}_\mu = \Theta^\alpha{}_\mu \tag{8}$$

Since

$$\tilde{R}^{\alpha\beta\lambda\sigma} = \frac{1}{4}\,\epsilon^{\alpha\beta\gamma\delta} R_{\gamma\delta\tau\rho} \epsilon^{\tau\rho\lambda\sigma} \tag{9}$$

then, by lowering and raising suffixes and contracting with $\alpha = \lambda$, we get

$$\tilde{R}^{\alpha\beta}{}_{\alpha\sigma} = \frac{1}{4}\,\epsilon^{\alpha\beta\gamma\delta} \epsilon_{\alpha\sigma\tau\rho} R^{\tau\rho}{}_{\gamma\delta} \tag{10}$$

and using the property

$$\epsilon^{\alpha\beta\gamma\delta}\epsilon_{\alpha\beta\tau\rho} = -2(\delta^{\gamma}_{\tau}\delta^{\delta}_{\rho} - \delta^{\gamma}_{\rho}\delta^{\delta}_{\tau}) \tag{11}$$

we are led to the components of Einstein's tensor

$$\tilde{R}^{\alpha\beta}{}_{\alpha\sigma} = -\frac{1}{2}\,\delta^{\beta}_{\sigma}R + R^{\beta}{}_{\sigma} = G^{\beta}{}_{\sigma} \tag{12}$$

Hence, by contraction ($\alpha = \mu$), the first of equations (8) leads to

$$\nabla^{\lambda}G_{\beta\lambda} = \Lambda_{\beta} \tag{13}$$

which violates the conservation law $\nabla^{\lambda}G_{\beta\lambda} = 0$, since $\Lambda^{\alpha}_{\beta\alpha} \neq 0$ in general. Thus, we conclude that Λ cannot be inserted into the first of equations (5) as a source for the rotational sector $\mathcal{R}$. However, if we consider $\Theta^{\alpha}_{\mu} = \kappa T^{\alpha}_{\mu}$, where κ is a constant, we see that the second of equations (8) becomes Einstein's equation

$$R^{\alpha}{}_{\mu} - \frac{1}{2}\,\delta^{\alpha}_{\mu}R = \kappa T^{\alpha}_{\mu} \tag{14}$$

if we choose κ as Einstein's constant. This means that Einstein's equation emerges from a break of dual symmetry of Bianchi's identity for the translational sector $\mathcal{T}$ of the $\mathcal{G}$-algebra. However, such a break cannot be taken for the rotational sector in the presence of a source, otherwise the conservation of Einstein's tensor is not satisfied.

A YM gauge Lagrangian is given in the form

$$\mathcal{L} = \overline{F} \wedge *\overline{F} \tag{15}$$

which can be written only for semisimple groups. In the Riemannian case this Lagrangian can be written for the Lorentz sector of the Poincaré group as

$$\mathcal{L} = R \wedge *R \tag{16}$$

and it leads to Yang's equation of gravitation (Yang, 1974)

$$R_{\mu\nu;\lambda} - R_{\mu\lambda;\nu} = 0 \tag{17}$$

This equation is only valid in the sourceless case and its physical significance has been studied exhaustively (see, for example, Thompson, 1975). The main point is that it contains solutions that do not lead to the correct relativistic perihelion shift of planets. From the geometrical point of view, the connection Γ can be interpreted as the gauge potential of Yang's theory of gravitation.

Concerning to the 2-form $\overline{F}$ pointed out before, we recall that in a four-dimensional spacetime manifold if we take the spacetime metric $g_{\mu\nu}$, its conformal transformation can be written in the form $\mathring{g}_{\mu\nu} = f(x) g_{\mu\nu}$. Since $g_{\mu\nu} g^{\mu\lambda} = \mathring{g}_{\mu\nu} \mathring{g}^{\mu\lambda} = \delta^{\lambda}_{\nu}$, we thus have

$$\mathring{g}^{\mu\lambda} = \frac{1}{f} g^{\mu\lambda} \tag{18}$$

In the diagonalized form, $|\det(\mathring{g}_{\mu\nu})| = f^4 \det(g_{\mu\nu})$, hence we obtain for the spacetime components of $*F$

$$\tilde{F} = \frac{\sqrt{g}}{2} g^{\mu\alpha} g^{\nu\beta} F_{\alpha\beta} \epsilon_{\mu\nu\lambda\sigma} \, dx^{\lambda} \wedge dx^{\sigma} \tag{19}$$

and for $*\mathring{F}$

$$\mathring{F} = \frac{(f^4 g)^{1/2}}{2} \frac{g^{\mu\alpha}}{f} \frac{g^{\nu\beta}}{f} \mathring{F}_{\alpha\beta} \epsilon_{\mu\nu\lambda\sigma} \, dx^{\lambda} \wedge dx^{\sigma} \tag{20}$$

Since $F_{\alpha\beta} = \mathring{F}_{\alpha\beta}$, because these components do not depend on spacetime metric, we conclude from equations (19) and (20) that $\tilde{F} = \mathring{F}$, which states the conformal invariance of $*F$. This means that the Lagrangian $\mathcal{L} = \overline{F} \wedge *\overline{F}$ is also conformally invariant, and the same is true for the YM equations. Moreover, we recall that Einstein–Hilbert Lagrangian of GR, $\mathcal{L} = g^{\mu\nu} R_{\mu\nu}$, is not of the YM type, which reinforces the fact that GR is not conformally invariant, as is already known (see, for instance, Fronsdal, 1984), and the curvature tensor does not behave as a gauge field of a YM gauge theory.

3. CONCLUDING REMARKS

The scenario developed here points out that Einstein's equations can emerge from a break of dual symmetry of Bianchi's identities for torsion. Such a break cannot be taken for curvature, otherwise the conservation of Einstein's tensor is not satisfied. Moreover, the approach proposed here yields a dynamical equation for the torsion field in the case of a general connection.

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